Approximate Bootstrap Technique*

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A "degenerate kernel" approximation technique is suggested for many-channel bootstrap problems. The approximation method reduces the solution of the matrix ND^{-1} integral equations to algebra and is especially suited for problems with complicated self-consistency constraints. It further avoids the subtraction-point dependence and lack of symmetry of the conventional "determinantal" approximation to the scattering matrix. The method is applied to the single-channel vector-meson bootstrap problem; the self-consistent solutions are discussed. Finally, it is shown that the scattering matrix obtained from the once subtracted matrix ND^{-1} integral-equation formalism is both symmetric and independent of the subtraction point.

I. INTRODUCTION

THE great majority of bootstrap calculations[1–12] that have been carried out to date have employed dynamical models based on the N/D method.[1] In most cases, moreover, an approximation due to Baker[13] (the first-order determinantal method) has been substituted for the full N/D integral-equation formulation. Now it is well known that the results of such approximations depend strongly upon the choice of subtraction point[12] and that they frequently bear little resemblance to the actual solutions of the N/D integral equations.[14] Furthermore, even those bootstrap calculations in which the integral equations were solved[8–10] have, of necessity, contained arbitrary parameters. This is because the driving forces have involved particles with spin $\geqslant 1$, with the consequence that the kernels of the integral equations are not Fredholm and a cutoff must be introduced to ensure the existence of solutions.

Turning to the many-channel bootstrap calculations,[3,6] we find an even less satisfactory situation. Although Bjorken[15] described the many-channel generalization of the N/D method shortly after the single-channel formalism was presented, its coupled sets of integral equations have never, to the author's knowledge, been employed in a dynamical calculation. Instead, a matrix version of the determinantal method has been used; it depends not only upon the choice of subtraction point, but also frequently leads to an unsymmetric scattering matrix, thus violating time reversal invariance. While the problem of "symmetry" can be overcome,[16] the symmetrized form of the determinantal method can lead, in somewhat special cases,[17] to unacceptable results. Furthermore, it cannot avoid the subtraction-point dependence.

The point of this discussion is that while the integral-equation formalism exists, it has either been prohibitively difficult to apply (as in many-channel bootstrap calculations) or has required the introduction of adjustable parameters, contrary to the bootstrap "philosophy." With regard to this last point, there is the encouraging possibility that the parameters of the Regge trajectories of the particles may be calculable in a self-consistent manner, so that the necessary "cutoffs" are provided without the introduction of arbitrary parameters. This program, which has not been successfully applied to date,[11] will not be discussed further here. Note, however, that the approximation technique to be described below is applicable to such a program.

Clearly, because the integral-equation formalism does exist, any approximation to that formalism should possess features that can justify its use. The first-order determinantal method is attractive because its application requires only the evaluation of integrals rather than the solving of integral equations. Another approximation, suggested by Fulton[18] and independently proposed by Shaw,[19] shares this feature and, furthermore, avoids the subtraction-point dependence and the lack of symmetry of the determinantal method. The Fulton-Shaw approximation, however, cannot be applied as casually as the determinantal method because it leads, in special cases, to unphysical results. In either case, the accuracy of the approximation cannot be

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13 M. Baker, Ann. Phys. (N.Y.) **4**, 271 (1958).
14 Compare the results for pion-nucleon scattering of Ref. 13 with those of Refs. 8 and 21, for example.
15 J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).
16 See, for example, A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963).
17 See footnote 14 of Ref. 3.
18 T. Fulton, in *Elementary Particle Physics and Field Theory, 1962 Brandeis Lectures* (W. A. Benjamin, Inc., New York, 1963), Vol. I, p. 55.
19 G. L. Shaw, Phys. Rev. Letters **12**, 345 (1964).

gauged without the explicit computation of the discontinuities of the approximate amplitude across the dynamic singularities. This tedious job is usually left undone in applications of these approximations.

The purpose of this note is to point out an approximation technique that retains the favorable features of the integral-equation formalism (independence of the choice of subtraction point and symmetry of the scattering matrix) while reducing the coupled sets of integral equations to algebraic equations. The method is based on the observation that a simple type of approximation to the "driving force" in the physical region reduces the kernels of the integral equations to degenerate kernels (or kernels of finite rank) and makes the solution of the equations trivial. Furthermore, the accuracy of the approximation is determined at the outset; it does not remain to be examined after the calculation is completed.

The details of the approximation technique are discussed in Sec. II. Section III contains the application of the technique to a simple single-channel problem, namely, the self-consistent generation of a vector-meson resonance in the elastic scattering of two pseudoscalar mesons. It is demonstrated that one simultaneously solves the problems of bootstrapping the ρ meson in pion-pion scattering and bootstrapping the degenerate vector-meson octet in the scattering of two degenerate pseudoscalar-meson octets. The self-consistent solutions obtained are presented graphically. Appendixes I and II are devoted to proving that the once-subtracted matrix ND^{-1} integral-equation formalism gives a scattering matrix that is both symmetric and independent of subtraction point. Appendix III contains the solutions of the integral equation for the vector-meson-bootstrap examples.

II. APPROXIMATION TECHNIQUE

The basis of the approximation technique to be described is Bjorken's matrix ND^{-1} formalism.[15] The symmetric partial-wave scattering matrix $T(z)$ for coupled two-particle channels is assumed to satisfy the unsubtracted dispersion relation

$$T(z)=B(z)+\frac{1}{\pi}\int\frac{dz'T^*(z')\rho(z')T(z')}{(z'-z)}\,,\tag{1}$$

where z represents an appropriate energy variable, the range of integration is over the physical branch cuts,[20] and $\rho(z)$ is a diagonal matrix of kinematic factors containing the standard step functions for the two-particle thresholds.[16] The symmetric matrix $B(z)$ represents the contributions of the dynamical singularities (the driving forces) and is assumed to be regular in the physical region. The unitarity condition represented by Eq. (1) may be conveniently written

$$\mathrm{Im}T^{-1}(z+i\epsilon)=-\rho(z)\tag{2}$$

[20] All integrals in this paper run over the physical branch cuts alone. The limits on the integrals have therefore been suppressed for the sake of simplicity.

for z in the physical region. Finally, the choice of definition of the scattering matrix elements, while of great importance in any given problem,[21] does not affect the present considerations. In the example of Sec. III, the appropriate choice will be described.

The standard procedure to obtain solutions of the nonlinear integral Eq. (1) is the ND^{-1} separation. The details of the separation, with the resultant linear inhomogeneous integral equations for $N(z)$ and $D(z)$, have been extensively discussed[22] and will not be repeated here. The convenient choice for the approximation technique under consideration is the integral equation for $N(z)$, which involves integrals only over the physical region. The resultant matrix equations in the coupled-channel case are[19]

$$T(z)=N(z)D^{-1}(z)\,,\tag{3}$$

$$N(z)=B(z)+\frac{1}{\pi}\int\frac{dx}{(x-z)}\times\left[B(x)-\frac{(z-z_0)}{(x-z_0)}B(z)\right]\rho(x)N(x)\,,\tag{4}$$

$$D(z)=1-(z-z_0)\frac{1}{\pi}\int\frac{dx\rho(x)N(x)}{(x-z_0)(x-z)}\,,\tag{5}$$

where z_0 is the subtraction point in the dispersion relation for $D(z)$. A once-subtracted form for $D(z)$ is necessary to provide convergence of the integrals in many problems of interest. It is shown in Appendix I that the scattering matrix $T(z)$ is independent of the choice of subtraction point. Thus, no additional parameter is introduced through this choice. Furthermore, Bjorken and Nauenberg[23] have shown (see also Appendix II) that the solution of Eqs. (3), (4), and (5), for symmetric $B(z)$, leads to a symmetric scattering matrix, thus satisfying time-reversal invariance.

The suggested approximation technique is based on the following observations. The kernel of the integral equation for $N(z)$ [Eq. (4)] is not singular at $x=z$ because the matrix quantity in square brackets

$$Q(x,z)=B(x)-\frac{(z-z_0)}{(x-z_0)}B(z)\tag{6}$$

vanishes for $x=z$. In other words, the integral in Eq. (4) is not a principal-value integral. Secondly, the kernel [call it $K(x,z)$] is not degenerate,[24] that is, cannot be written

$$K(x,z)=\sum_{i=1}^{n}F_i(z)G_i(x)\,,\tag{7}$$

[21] See A. W. Martin and J. L. Uretsky, Phys. Rev. **135**, B803 (1964) for a discussion of the limitations involved in such a choice.
[22] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).
[23] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).
[24] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. I, Chap. III.

where n is finite, only because of the factor $(x-z)^{-1}$. Finally, the knowledge of $B(z)$ in the physical region alone determines $N(z)$ in the physical region and $D(z)$ everywhere. And $B(z)$ is a matrix of *regular* functions in the physical region.

The approximation technique is now obvious. If the elements of $B(z)$ in the physical region are approximated by any ratios of polynomials in z, then the elements of $Q(x,z)$ [Eq. (6)] may be written as ratios of polynomials in x and z. Because $Q(x,z)$ must vanish for $x=z$, the common factor $(x-z)$ may be extracted from $Q(x,z)$ and cancels the term $(x-z)^{-1}$. As a result, the kernel of the integral equation is degenerate and the solution of the equation reduces to algebra. It is also evident that the twin features of symmetry and independence of the subtraction point are retained by the approximation technique. This is because it provides not so much an approximate solution to exact equations as an exact solution to approximate equations.

With regard to practical applications of the method, the reader will recall that the partial-wave driving forces are usually characterized by logarithmic factors. The asymptotic behavior of these factors cannot be matched by simple ratios of polynomials. On the other hand, it is this asymptotic behavior in the vector-meson bootstrap problem,[10] for instance, that prevents the solution of the integral Eq. (4) and requires the introduction of a cutoff. In the treatment of this problem in the following section, the cutoff is imposed by replacing the logarithmic term with a constant. A different situation is encountered in meson-baryon scattering problems with single-baryon-exchange forces. Here, the logarithmic terms are unimportant in the asymptotic behavior and simple ratios of polynomials can provide quite satisfactory approximations. The technique then offers an excellent vehicle for coupled-channel analyses of the type considered by Martin and Wali[16] and, more recently, by Wali and Warnock.[25]

Two distinct approaches to applications of the approximation become evident. The crux of the method, of course, is the conversion of the kernel to degenerate form. This conversion, however, is independent of the inhomogeneous term in the integral equation [the first term on the right in Eq. (4)]. One therefore has the choice (as long as all of the integrals converge) of either keeping the "exact" driving force for the inhomogeneous term or using the same approximation as that employed in the kernel. In either case the solution of the integral equation reduces to algebra. The drawback in the use of the "exact" form is that the solution usually will no longer be symmetric or independent of the subtraction point.

The proof of this is trivial because all that is required is a counter example. Let the inhomogeneous term be $B(z)+C(z)$, where the symmetric matrix $C(z)$ represents the difference between the approximation $B(z)$ and the exact form. Now let $B(z)$ tend towards zero. The solution of Eqs. (3), (4), and (5) then tends toward the first-order determinantal approximation, which is well known to be neither symmetric nor independent of subtraction point. Of course, if the approximation employed is "good enough," these deviations will be minor and it may well be true that the solution with the exact inhomogeneous term provides the better approximation to the solution of the original integral equation.

While it is clear that this approximation technique cannot be a fully adequate substitute for solving the unmodified integral equation, it does offer some striking computational advantages in bootstrap calculations. In addition to the primary feature of reducing the solution to algebra, the method has the advantage that (in the case in which the inhomogeneous term is approximated) the integrals encountered in the solution are often trivial. In the search for self-consistent solutions, by computer for example, this fact can represent an enormous saving of time.

Finally, the adequacy of the approximation is determined by the user—not by the formalism as it is with the determinantal method. Also, because this "degenerate kernel" approximation can be free (in principle) of arbitrary parameters, it offers the possibility of exploring the consequences of the bootstrap idea in a significantly simplified way.

III. VECTOR-MESON BOOTSTRAP PROBLEM

As an illustration of the "degenerate kernel" approximation described in the preceding section, a simple single-channel bootstrap problem has been studied. It is the familiar self-consistent generation of a vector-meson resonance in the elastic scattering of two pseudoscalar mesons, historically the first of the bootstrap models.[1,2] Because of an amusing coincidence, it turns out that one simultaneously solves the following two problems: the bootstrap of the ρ meson in pion-pion scattering, and the bootstrap of the vector-meson octet in the elastic scattering of two pseudoscalar-meson octets in the limit of exact unitary symmetry.[26] This "degeneracy" will be discussed in full below.

Consider first the partial-wave amplitudes $A_l(s)$, which are assumed to be analytic in the cut s plane, and satisfy the elastic unitarity condition

$$A_l(s)=[s/(s-4\mu^2)]^{1/2}\sin\delta_l\exp(i\delta_l)\quad(\delta_l\text{ real})\qquad(8)$$

in the physical region ($s\geqslant 4\mu^2$). Here, s is the square of the total energy in the barycentric system, and μ is the mass of the pseudoscalar mesons.[27] In keeping with the simplest form of the bootstrap idea, the driving force for these partial-wave amplitudes is limited to single-vector-meson exchange in the crossed channels. Attention will also be restricted to the p-wave amplitudes

[25] K. C. Wali and R. L. Warnock (to be published).

[26] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).

[27] The natural units $\hbar=c=1$ are employed.

$(l=1)$ so that henceforth the partial-wave subscript will be dropped. In this one-particle-exchange approximation, the p-wave driving force is well known to be

$$\hat{B}(s)=\frac{\alpha(s-2\mu^2+\frac{1}{2}v^2)}{2(s-4\mu^2)}\times\left[\left(1+\frac{2v^2}{s-4\mu^2}\right)\ln\left[1+\frac{s-4\mu^2}{v^2}\right]-2\right], \quad (9)$$

where v is the (real) mass of the exchanged vector meson and α represents the appropriate dependence on the isotopic spin factors and coupling constants. The fact that the self-consistent mass of the vector meson will turn out to be complex is ignored for the sake of simplicity.

Because the driving force is limited to single-particle exchange, and therefore contains the correct threshold zeros, the amplitude obtained from the N/D integral-equation formalism will not possess the required threshold zeros unless the amplitudes are redefined so that they are nonvanishing at threshold. To this end, a new p-wave amplitude and driving force are defined by the expressions

$$T(s)\equiv[4\mu^2/(s-4\mu^2)]A(s),$$
$$B(s)\equiv[4\mu^2/(s-4\mu^2)]\hat{B}(s), \quad (10)$$

and it is convenient to introduce[5] the dimensionless energy variable $z=s/4\mu^2$ and the mass ratio $r=v^2/4\mu^2$.

The p-wave amplitude $T(z)$ of Eq. (10) is assumed to satisfy the unsubtracted dispersion relation

$$T(z)=B(z)+\frac{1}{\pi}\int_1^\infty\frac{dz'\rho(z')|T(z')|^2}{(z'-z)}, \quad (11)$$

where the kinematic factor $\rho(z)$ arising from the elastic unitarity condition is

$$\rho(z)=(z-1)^{3/2}/z^{1/2}, \quad (12)$$

and, from Eqs. (9) and (10), the driving force $B(z)$ is

$$B(z)=\frac{\alpha(z+\frac{1}{2}r-\frac{1}{2})}{2(z-1)^2}\left[\left(1+\frac{2r}{z-1}\right)\ln\left[1+\frac{z-1}{r}\right]-2\right]. \quad (13)$$

It is a trivial matter to verify that the dispersion relation (11) satisfies the requirements of the Phragmen-Lindelöf theorem[21] and that the choice of amplitude in Eq. (10) is therefore an acceptable one.

In order to determine the appropriate coefficients α for the pion-pion and degenerate octet-model problems (hereafter called the π-π-ρ and P-P-V problems for brevity), the following coupling-constant notation is used. Let the pure F-type coupling between two pseudoscalar-meson octets and the vector-meson octet be defined so that the coupling of the ρ meson to two pions is written $g_{\rho\pi}\boldsymbol{\rho}_\mu\cdot\boldsymbol{\pi}\times d^\mu\boldsymbol{\pi}$ in the conventional isotopic-spin representation. The couplings between the other pseudoscalar mesons and the vector mesons are then uniquely defined in terms of the π-π-ρ coupling constant $g_{\rho\pi}$. For the single-channel π-π-ρ problem, the same effective coupling will be employed.[28]

Taking proper account of the identicality of the pions, we find the appropriate coefficient α for the isospin-1 channel in the π-π-ρ problem to be

$$\alpha=g_{\rho\pi}^2/4\pi. \quad (14)$$

Similarly, in the octet-model problem, the coefficient for the antisymmetric eightfold representation (the representation to which the vector mesons are assigned) turns out to be

$$\alpha=\tfrac{3}{2}g_{\rho\pi}^2/4\pi. \quad (15)$$

The self-consistency requirement demands that the solution of Eq. (11), with the driving force of Eq. (13), should exhibit a resonance or bound state at the position $s=v^2$(or $z=r$) with a width (or residue in the bound-state case) that is linearly related to the coupling constant $g_{\rho\pi}^2/4\pi$. The explicit relationship between the width and coupling constant for the two problems of interest is readily derived by means of the "sharp-resonance" approximation and the crossing relations. The result for the π-π-ρ problem is

$$g_{\rho\pi}^2/4\pi=[6r/(r-1)^{3/2}](\Gamma/\mu), \quad (16)$$

while that for the P-P-V problem turns out to be

$$g_{\rho\pi}^2/4\pi=[4r/(r-1)^{3/2}](\Gamma/\mu), \quad (17)$$

where Γ in both (16) and (17) is the full width at half-maximum of the resonance.

The amusing coincidence referred to earlier is now evident. With the aid of Eqs. (14) and (15), both (16) and (17) may be written as

$$\alpha=[6r/(r-1)^{3/2}](\Gamma/\mu). \quad (18)$$

It follows that the self-consistent solution for the variables r and α must be the same for both the π-π-ρ and P-P-V problems, although the interpretation of the magnitude of the coupling constant and the total energy of the resonance will clearly differ between the two problems. The constraints of the self-consistency requirement have been stated. It remains to apply the degenerate-kernel approximation to the problem and obtain the self-consistent solutions.

The asymptotic behavior of the driving force $B(z)$ [Eq. (13)] for large z is seen to be

$$B(z)\to(\alpha/2z)\ln(z), \quad z\to+\infty, \quad (19)$$

and it is the $\ln(z)$ factor that prevents the solution of the N/D integral equation when the "exact" driving force is employed. In applying the degenerate-kernel approximation, the $\ln(z)$ term in Eq. (19) will be re-

[28] Note that $g_{\rho\pi}$ as defined here is related to $\gamma_{\rho\pi\pi}$ in the notation of Refs. 2, 3, and 5 by $g_{\rho\pi}=2\gamma_{\rho\pi\pi}$.

placed by a constant. This replacement provides the necessary "cutoff" in the problem. The purpose of the first approximation to be discussed was to study the dependence of the self-consistent solutions on the value of this cutoff constant.

The driving force of Eq. (13) was approximated by the function

$$B(z)=\frac{\alpha\beta(z+\frac{1}{2}r-\frac{1}{2})}{2(z+r-1)(z+6\beta r-1)}, \tag{20}$$

which has the correct threshold value, closely reproduces the first few derivatives at threshold (especially for large β), and tends asymptotically as

$$B(z)\to\alpha\beta/2z\,,\quad z\to+\infty\,. \tag{21}$$

Let us attempt to guess a "reasonable" value for the cutoff constant β by supposing that the cutoff should become effective somewhere above the first inelastic threshold at, say, $s\approx 24\mu^2$ or $z\approx 6$. This rough estimate suggests that $\beta\gtrsim 2$ might be a reasonable starting point. It was discovered, however, that no self-consistent solutions exist for β in this range.

The partial-wave amplitude obtained from the approximate driving force of Eq. (20) (which is readily seen to be a two-pole approximation) is stated in Appendix III. The self-consistent solutions for the mass ratio r and the coupling constant α, as a function of the cutoff parameter β, are plotted in Figs. 1 and 2, respectively. Two features of the problem are thought worthy of comment. First, "resonant" solutions exist only for $\beta\lesssim 0.6$, which represents a surprisingly low cutoff energy. Second, no "bound-state" solutions were obtained. For values of the mass ratio $r<1.0$ (i.e., for vector-meson bound states), it was found impossible to satisfy the self-consistency requirement on the coupling constant with the approximation of Eq. (20).[29]

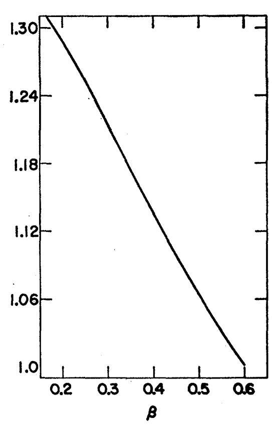

FIG. 1. Self-consistent solutions for the mass ratio r as a function of the parameter β in the two-pole approximation.

[29] The reason no bound-state solutions were obtained is that the direct vector-meson pole term (the bound-state pole) is not included in the driving force $B(z)$. If this term were included, the self-consistent solutions would continue smoothly into the "bound-state" region. The author wishes to thank Professor C. Goebel for a discussion of this point.

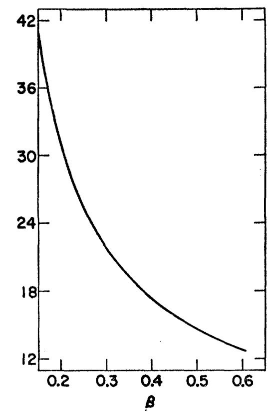

FIG. 2. Self-consistent solutions for the coupling constant α as a function of the parameter β in the two-pole approximation.

Because of the small values of the cutoff constant required for the existence of resonant solutions, and the fact that Eq. (20) does not provide a good approximation for such values of β, it was felt necessary to repeat the calculation with a more adequate approximation. One possibility, suggested by Childers,[30] is the expression

$$B(z)=\frac{\alpha\gamma[2(r+1)+\gamma(r-1)]}{24r(z+\gamma r-1)}-\frac{\alpha\gamma^2[r+1+\gamma(r-1)]}{24(z+\gamma r-1)^2}, \tag{22}$$

which reproduces the threshold value and slope of Eq. (13) for all values of γ. Note that for $\gamma=2$ Childers' approximation corresponds to expanding the logarithmic term in $B(z)$ [Eq. (13)] according to

$$\ln(1+x)=2\left[\left(\frac{x}{x+2}\right)+\tfrac{1}{3}\left(\frac{x}{x+2}\right)^3+\tfrac{1}{5}\left(\frac{x}{x+2}\right)^5+\cdots\right] \tag{23}$$

and keeping only those terms in $B(z)$ that are nonvanishing at threshold.

The solution of the integral equation (4) for the "single-pole double-pole" approximation of Eq. (22) is given in Appendix III. The self-consistent solutions for r and α, as a function of the parameter γ, are plotted in Figs. 3 and 4, respectively. It is evident that the solutions obtained with the two approximate driving forces are quite similar; it is of interest to note that the asymptotic limits of the approximations (20) and (22), which lead to the same self-consistent value of r, are almost identical. This fact indicates that the asymptotic behavior of the driving force may play a crucial role in determining the nature of the bootstrap solutions. That is, the self-consistent values of r and α may depend sensitively on the form of cutoff employed. This possibility makes it necessary, in the author's opinion, to pursue the goal of calculating the parameters of the

[30] R. W. Childers (private communication).

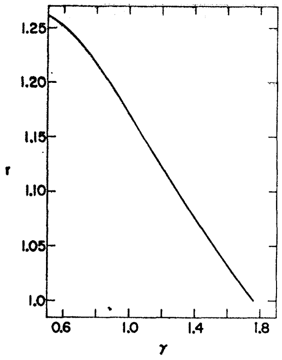

FIG. 3. Self-consistent solutions for the mass ratio r as a function of the parameter γ in the single-pole double-pole approximation.

Regge trajectories of the particles in a self-consistent manner.[11,31]

It is also of interest to compare the results of the degenerate-kernel approximation with those of the determinantal method. Fixing the subtraction point at the beginning of the "left-hand" cut ($z_0=1-r$) leads to the self-consistent determinantal solution[32] $r=1.47$ and $\alpha=10.35$. In the π-π-ρ problem this solution corresponds to a ρ meson with a total energy of 337 MeV and a full width at half maximum of 53 MeV. In the P-P-V problem, on the other hand, if the degenerate pseudoscalar-meson octet is taken to have the mass of the K meson, the determinantal solution gives a degenerate vector-meson octet with total energy 1200 MeV and full width 187 MeV.

Of course, with regard to the octet-model interpretation, the vector mesons and pseudoscalar mesons are not degenerate. It is the author's opinion that a more reasonable interpretation follows from the observation that in the bootstrap problem for the nondegenerate vector mesons,[6] the bootstrap of the isospin-zero member of the octet (the ϕ meson, say) remains a single-channel (the K-$\bar{K}$ channel) problem. Retaining the assumption of degeneracy only in the driving forces, then, allows us to interpret the results of these calculations in terms of the pseudoscalar K meson and the vector ϕ meson. For amusement, let us adjust the parameter γ in Childers' approximation [Eq. (22)] so that the self-consistent mass ratio r corresponds exactly to the experimental ϕ-meson/K-meson ratio, i.e., to $r=1.06$. The resulting prediction for the full width of the ϕ meson is 17 MeV, which is substantially larger than the observed width.

In this simple analysis the important questions of ω-ϕ mixing, the existence of other vector-meson octets, etc., have been ignored. The present calculation is much too crude to be applied to such questions. General features of the self-consistent solutions, however, may hold true in more sophisticated treatments. For example, it is evident from the solutions obtained that a satisfactory π-π-ρ solution (one for which $r\approx 7.3$ and $\alpha\approx 2.0$) is nowhere in sight. This is undoubtedly due to the fact that in pion-pion scattering the ρ meson is a "high-energy" phenomenon and the single-channel approach with elastic unitarity is therefore an inadequate approximation.

From the point of view of the octet model, on the other hand, the vector mesons may more reasonably be considered a "low-energy" phenomenon in the elastic scattering of pseudoscalar mesons. Certainly, the solutions obtained in the present calculation make much more sense in the octet-model interpretation. The author does not want to give the impression that he attaches any great weight to the results of this simple calculation; the fact that an effective cutoff had to be introduced precludes such a feeling. Rather, the calculation is intended as an example of the ease with which self-consistency problems can be handled in the degenerate-kernel approximation.

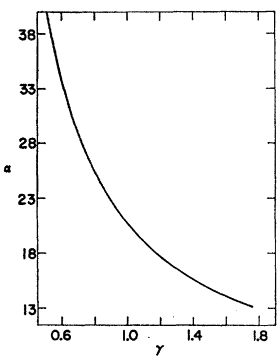

FIG. 4. Self-consistent solutions for the coupling constant α as a function of the parameter γ in the single-pole double-pole approximation.

IV. CONCLUSIONS

A "degenerate kernel" approximation technique has been presented. The method is equally applicable to single-channel and many-channel problems; but it is in the many-channel case that the approximation technique should prove most useful, especially when self-consistency constraints are imposed. The approximation procedure is based on the ND^{-1} integral-equation formalism and retains the features of that formalism, namely, the subtraction-point independence and the symmetry of the scattering matrix. In this way it avoids the principal drawbacks of the conventional determinantal approximation.

The crux of the approximation technique is the conversion of the kernel of the integral equation to degenerate form. This is accomplished through the approximation of the elements of the "driving-force" matrix, *in the physical region alone*, by arbitrary ratios of polynomials. This is equivalent, of course, to representing the contribution of the dynamical singularities (the "left-hand" cuts) by an arbitrary number of poles, and the approximation can be made, in general, as

[31] S. C. Frautschi, P. E. Kaus, and F. Zachariasen, Phys. Rev. **133**, B1607 (1964).

[32] See Refs. 2, 3, 5, and 12. The solution quoted is the result of an independent calculation by the author.

accurate as desired. Once the kernel is converted to degenerate form, the solution of the integral equation becomes an algebraic problem. This is true whether or not the inhomogeneous term in the integral equation is also approximated.

Although it has been shown that when the "exact" inhomogeneous term is kept (and only the kernel is approximated), the scattering matrix obtained will not usually be symmetric or independent of the subtraction point; it nevertheless seems likely that this form of the degenerate-kernel approximation will provide the superior approximation to the solution of the original integral equation. For very complicated problems, on the other hand, the ease of application of the "full" approximation procedure (based on the fact that the integrals encountered are usually trivial) may well offer the decisive consideration.

For the purpose of illustration, the degenerate-kernel approximation has been applied to the single-channel vector-meson bootstrap problem. The primary result of the calculation is the indication that the self-consistent solutions depend sensitively upon the asymptotic behavior of the driving force. In other words, the solutions are apparently quite dependent on the cutoff. This indication emphasizes the need for self-consistent determinations of the Regge trajectories of the particles so that the necessary cutoffs will be "built into" the calculation. A secondary result of the analysis is the suggestion that the interpretation of the bootstrap problem will follow more readily from the octet-model point of view. That is, the substantial difference between the masses of the ρ meson and the pion makes it difficult, in the π-π-ρ problem, to attach any significance to the "low-energy" approximations employed here.

Finally, the approximation technique described in this paper is presently being applied to the nondegenerate vector-meson bootstrap problem,[6] within the octet-model framework, in order to determine the self-consistent deviations of the coupling constants and vector-meson masses from the exact unitary-symmetry predictions. The results will be described in a forthcoming paper.

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APPENDIX I: SUBTRACTION POINT INDEPENDENCE

It is widely known that the scattering matrix $T(z)$, defined through Eq. (3) and the solution of Eqs. (4) and (5), is independent of the choice of subtraction point. To the author's knowledge, however, no proof of this fact has appeared in the literature. In order to fill this minor gap, a simple proof (omitting a number of algebraic steps) is presented here.

Let us emphasize the apparent subtraction-point dependence by rewriting Eqs. (3), (4), and (5) in the form

$$T(z,z_0)=N(z,z_0)D^{-1}(z,z_0)\,, \tag{I.1}$$

$$N(z,z_0)=B(z)+\frac{1}{\pi}\int\frac{dx}{(x-z)}\times\left[B(x)-\frac{(z-z_0)}{(x-z_0)}B(z)\right]\rho(x)N(x,z_0)\,, \tag{I.2}$$

$$D(z,z_0)=1-(z-z_0)\frac{1}{\pi}\int\frac{dx\rho(x)N(x,z_0)}{(x-z_0)(x-z)}\,. \tag{I.3}$$

Assuming that a solution exists to (I.2) and (I.3), we may write the identity

$$D^{-1}(z_1,z_0)=1+(z_1-z_0)\frac{1}{\pi}\int\frac{dx\rho(x)N(x,z_0)D^{-1}(z_1,z_0)}{(x-z_0)(x-z_1)} \tag{I.4}$$

and note that the matrix $N(z,z_1)\equiv N(z,z_0)D^{-1}(z_1,z_0)$ satisfies the integral equation (I.2) with z_0 replaced by z_1. It follows that $N(z,z_1)$ is independent of z_0. Similarly, the matrix $D(z,z_1)\equiv D(z,z_0)D^{-1}(z_1,z_0)$ satisfies (I.3) with z_0 replaced by z_1 and is independent of z_0. The proof is completed by writing (I.1) as

$$\begin{aligned}T(z,z_0)&=N(z,z_0)D^{-1}(z_1,z_0)[D(z,z_0)D^{-1}(z_1,z_0)]^{-1}\\&=N(z,z_1)D^{-1}(z,z_1)=T(z,z_1)\,.\end{aligned} \tag{I.5}$$

The scattering matrix $T(z,z_0)$ is therefore unaffected by variations in the subtraction point.

APPENDIX II: SYMMETRY OF SCATTERING MATRIX

Bjorken and Nauenberg[23] first proved that the scattering matrix defined by Eq. (3) is symmetric provided that its "left-hand" discontinuities are symmetric. Their concise proof, however, rested heavily on the assumed asymptotic behavior of the scattering matrix $T(z)$ and does not necessarily apply to the once-subtracted ND^{-1} formalism of Eqs. (4) and (5). It is the purpose of this appendix to present a proof that is independent of the asymptotic behavior of the scattering matrix but which is limited, at the same time, to the once-subtracted formalism.

Because the subtraction point plays a central role in the proof, the notation of Appendix I will be employed. It is clear from Eqs. (I.1) and (I.3) that

$$D(z_0,z_0)=1\,,\quad T(z_0)=N(z_0,z_0)\,, \tag{II.1}$$

where the revised notation for the scattering matrix reflects the fact that $T(z)$ is independent of the subtraction point. We now concentrate on proving that $N(z_0,z_0)$ is symmetric.[33] From Eq. (I.2) follows the

[33] The author is indebted to Dr. W. D. McGlinn for suggesting this approach to the proof.

identity

$$N(z_0,z_0)=B(z_0)+\frac{1}{\pi}\int\frac{dxB(x)\rho(x)N(x,z_0)}{(x-z_0)}. \quad \text{(II.2)}$$

Since $B(z)=B^T(z)$ and $\rho(z)=\rho^T(z)$, where the superscript T denotes the transpose, the first term on the right-hand side of (II.2) is symmetric and we need only consider the matrix

$$F(z_0)=\int\frac{dxB(x)\rho(x)N(x,z_0)}{(x-z_0)}. \quad \text{(II.3)}$$

The transpose of Eq. (I.2) may be written

$$B(x)=N^T(x,z_0)-\frac{1}{\pi}\int\frac{dyN^T(y,z_0)\rho(y)}{(y-x)} \times\left[B(y)-\frac{(x-z_0)}{(y-z_0)}B(x)\right]. \quad \text{(II.4)}$$

Substituting (II.4) into (II.3) gives

$$F(z_0)=\int\frac{dxN^T(x,z_0)\rho(x)N(x,z_0)}{(x-z_0)} -\frac{1}{\pi}\int\int\frac{dxdyN^T(y,z_0)\rho(y)}{(x-z_0)(y-z_0)(y-x)} \times[(y-z_0)B(y)-(x-z_0)B(x)]\rho(x)N(x,z_0). \quad \text{(II.5)}$$

The first term on the right in (II.5) is obviously symmetric. The second term is also symmetric as may immediately be verified by writing its transpose and interchanging the dummy integration variables x and y. It follows that $N(z_0,z_0)$ is symmetric and, by virtue of (II.1), $T(z_0)$ is symmetric.

The proof is completed by recalling that $T(z)$ is independent of the subtraction point. That is, the transformation of Eq. (I.5) may be employed to prove that $T(z)$ is symmetric for z equal to any allowed value of the subtraction point, namely everywhere on the real axis apart from the physical cuts. Because the scattering matrix is symmetric on an interval, its analytic continuation is symmetric in the entire plane.

APPENDIX III: SPECIAL SINGLE-CHANNEL SOLUTIONS

In the single-channel application (Sec. III) of the degenerate-kernel approximation, two approximations to the driving force were employed. The purpose of this appendix is to list the simplest forms for the scattering amplitude obtained from the solutions of the integral equation (4) for the two approximations. In each case the single-channel amplitude is represented by $T(z)=N(z)/D(z)$ and all integrals run only over the physical branch cut.

For the first case, the two-pole approximation, the driving force $B(z)$ is

$$B(z)=R_1/(z-c_1)+R_2/(z-c_2). \quad \text{(III.1)}$$

The solution is then given by

$$N(z)=R_1(1-K_1)/(z-c_1)+R_2(1-K_2)/(z-c_2),$$

$$D(z)=1-R_1(1-K_1)(z-c_1)\frac{1}{\pi}\int\frac{dx\rho(x)}{(x-c_1)^2(x-z)} -R_2(1-K_2)(z-c_2)\frac{1}{\pi}\int\frac{dx\rho(x)}{(x-c_2)^2(x-z)}-K_1K_2, \quad \text{(III.2)}$$

where

$$K_1=R_2(c_1-c_2)\frac{1}{\pi}\int\frac{dx\rho(x)}{(x-c_1)(x-c_2)^2}, \quad \text{(III.3)}$$

and K_2 is obtained from (III.3) through the interchange of the subscripts 1 and 2.

For the second case, the single-pole double-pole approximation, the driving force $B(z)$ is

$$B(z)=a/(z-c)+b/(z-c)^2. \quad \text{(III.4)}$$

The solution is then given by

$$N(z)=k/(z-c)+b/(z-c)^2,$$

$$D(z)=1-k(z-c)\frac{1}{\pi}\int\frac{dx\rho(x)}{(x-c)^2(x-z)} -b(z-c)\frac{1}{\pi}\int\frac{dx\rho(x)}{(x-c)^3(x-z)}, \quad \text{(III.5)}$$

where

$$k=[a-(b^2/\pi)\int dx\rho(x)/(x-c)^4] \times[1+(b/\pi)\int dx\rho(x)/(x-c)^3]^{-1}. \quad \text{(III.6)}$$

It is a simple matter to verify that the amplitudes obtained from (III.2) and (III.5) are unitary in the physical region, possess the specified dynamic singularities, and clearly contain no dependence on an arbitrary subtraction point. Additionally, for the kinematic factor $\rho(x)$ of Eq. (12), the integrals are trivial and the self-consistent solutions of Sec. III may readily be obtained.